A wide family of singularity-free cosmological models

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In this paper a family of non-singular cylindrical perfect fluid cosmologies is derived. The equation of state corresponds to a stiff fluid. The family depends on two independent functions under very simple conditions. A sufficient condition for geodesic completeness is provided.

I. INTRODUCTION

After the discovery of the first regular perfect fluid cosmological model by Senovilla [1], one of the major questions that that spacetime posed was to determine how generic this lack of singularities was. For instance, in the Ruiz-Senovilla [2] barotropic family with equation of state $p = \gamma\mu$, $0 < \gamma < 1$, the regular models were those corresponding to a radiation fluid, $\gamma = 1/3$. These solutions are the separatrix between those models with singular pressure and energy density, $\gamma > 1/3$, and those who just had singular Weyl curvature, $\gamma < 1/3$ [3]. Therefore regular models would be a zero-measure set in this family.

Our aim in this paper is to determine an infinite family of regular cosmological models that need not be so restrictive as the Ruiz-Senovilla family and therefore may indicate that regular models cannot be neglected in the set of solutions of Einstein equations.

We shall focus on stiff perfect fluids, since they are simple enough to allow almost complete integration of Einstein equations and thereby constitute an excellent arena for checking hypotheses. We shall show that under very simple restrictions regular solutions appear.

The second section of this paper will be devoted to deriving solutions of the stiff fluid Einstein equations in a convenient manner for our purposes. In the third section geodesic completeness of the solutions shall be imposed and the restrictions derived from this assumption shall be expressed as a sufficient condition.

II. STIFF FLUID COSMOLOGIES

We shall restrict to spacetimes endowed with an Abelian orthogonally transitive group of isometries, G_2, acting on timelike surfaces, since this is the framework where regular cosmological models have so far appeared in. We shall further require that the Killing fields be mutually orthogonal. Adapting the coordinates to these fields we shall write them as $\{\partial_z, \partial_\phi\}$. Under these assumptions we can write the metric for the spacetime in a convenient coordinate chart [4], $\{t, r, z, \phi\}$,

$$ds^2 = e^{2K}(-dt^2 + dr^2) + e^{-2U}dz^2 + \rho^2 e^{2U}d\phi^2, \tag{1}$$

which has been chosen as isotropic on the non-ignorable coordinates, t, r. This is a priori no restriction and may always be achieved since every 2-metric admits an isotropic parametrization. The metric is written in terms of three

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functions, K, U and ρ that depend only on t and r.

We may interpret the isometry group as cylindrical symmetry in the spacetime provided we have a regular axis where the norm of the angular Killing field vanishes. We shall come back to this issue later on. The range of the coordinates will be then the usual one for cylindrical symmetry,

$$-\infty < t, z < \infty, \ 0 < r < \infty, \ 0 < \phi < 2\pi. \tag{2}$$

The matter content of the spacetime will be a perfect fluid of energy density μ, pressure p and 4-velocity u. The energy-momentum tensor will be then,

$$T^{\mu\nu} = \mu u^\mu u^\nu + p\,(g^{\mu\nu} + u^\mu u^\nu), \qquad 0 \le \mu, \nu \le 3, \qquad u^\mu u_\mu = -1. \tag{3}$$

For a stiff fluid, $\mu = p$. We shall write down the Einstein equations, $R_{\mu\nu} - R\,g_{\mu\nu}/2 = T_{\mu\nu}$, in a comoving system of coordinates for the perfect fluid, that is, $u = e^{-K}\partial_t$. After some simplications the equations read,

$$U_{tt} - U_{rr} + \frac{1}{\rho}(U_t\rho_t - U_r\rho_r) = 0, \tag{4a}$$

$$\rho_{tt} - \rho_{rr} = 0, \tag{4b}$$

$$K_t\rho_r + K_r\rho_t = \rho_{tr} + U_t\rho_r + U_r\rho_t + 2\rho U_t U_r, \tag{4c}$$

$$K_t\rho_t + K_r\rho_r = \frac{\rho_{tt} + \rho_{rr}}{2} + U_t\rho_t + U_r\rho_r + \rho(U_t^2 + U_r^2) + p\rho e^{2K}, \tag{4d}$$

$$K_{rr} - K_{tt} + \frac{U_r\rho_r - U_t\rho_t}{\rho} + U_r^2 - U_t^2 = pe^{2K}, \tag{4e}$$

$$K_r + \frac{p_r}{2p} = 0, \tag{4f}$$

$$K_t + \frac{\rho_t}{\rho} + \frac{p_t}{2p} = 0, \tag{4g}$$

where the last pair of equations are just Euler and continuity equations for the perfect fluid.

Since every regular cosmological model in the literature has a ρ with spacelike gradient, we shall impose as an ansatz that $\operatorname{grad}\rho$ be orthogonal to the velocity of the fluid, u.

Under this assumption, ρ will be a function of r only. But then (4b) requires that ρ be a linear function of r. After rescaling the coordinates we can take $\rho = r$ and the whole system of equations becomes rather simple,

$$U_{tt} - U_{rr} - \frac{U_r}{r} = 0, \tag{5a}$$

$$K_t = U_t + 2rU_t U_r, \tag{5b}$$

$$K_r = U_r + r(U_t^2 + U_r^2) + pre^{2K}, \tag{5c}$$

$$K_{rr} - K_{tt} + \frac{U_r}{r} + U_r^2 - U_t^2 = pe^{2K}, \tag{5d}$$

$$K_r + \frac{p_r}{2p} = 0, \tag{5e}$$

$$K_t + \frac{p_t}{2p} = 0. \tag{5f}$$

The energy-momentum conservation equations can be integrated,

$$p = \alpha e^{-2K}, \tag{6}$$

with $\alpha = \text{const.} > 0$ and equation (5d) is a consequence of the others. We are left then with a two-dimensional reduced wave equation in polar coordinates without source term (5a) and a quadrature for K,

$$K_t = U_t + 2rU_t U_r, \tag{7a}$$

$$K_r = U_r + r(U_t^2 + U_r^2) + \alpha r, \tag{7b}$$

that can be integrated after providing a solution to the wave equation. The integrability condition for the quadrature (7) is namely the wave equation, so the whole problem reduces to solving it.

Following [5], for instance, the solution to the Cauchy problem for the wave equation in the plane can be constructed from the $3D$ solution by ignoring the third variable,

$$U(x,y,t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^t dR R \frac{g(x + R\cos\phi, y + R\sin\phi)}{\sqrt{t^2 - R^2}} + \frac{1}{2\pi} \frac{\partial}{\partial t} \int_0^{2\pi} d\phi \int_0^t dR R \frac{f(x + R\cos\phi, y + R\sin\phi)}{\sqrt{t^2 - R^2}}, \quad (8)$$

for initial data $U(x,y,0) = f(x,y)$, $U_t(x,y,0) = g(x,y)$. Note that the $2D$ wave equation does not satisfy Huygens' principle and therefore the domain of dependence is a circle, not just a circumference.

Our problem is even easier since we do not have dependence on the polar angle. Therefore we just have to impose circular symmetry on the initial data $U(r,0) = f(r)$, $U_t(r,0) = g(r)$. The time coordinate may be removed from the integration limits by an appropriate scaling, $R = t\tau$,

$$U(r,t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^1 d\tau \frac{\tau}{\sqrt{1-\tau^2}} \left\{ tg(v) + f(v) + tf'(v)\frac{t\tau^2 + r\tau\cos\phi}{v} \right\}, \quad (9)$$

where $v = \sqrt{r^2 + t^2\tau^2 + 2rt\tau\cos\phi}$, choosing the origin of the polar angle at the angle for (x,y). This expression is valid for all values of t.

For instance, the non-singular spacetime in [6] is generated by

$$f(x) = \frac{\beta}{2}x^2, \qquad g(x) = 0, \qquad \beta > 0.$$

It is clear now that we are just integrating the functions f,g on a finite interval of r and that the integrals are well-defined provided g, f, f' are continuous. The singularity at $\tau = 1$ is harmless under such conditions.

The solution does not share the class of differentiability of the initial data because of the derivative term. We need at least $f \in C^3([0,\infty))$, $g \in C^2([0,\infty))$ in order to have $U, K \in C^2([0,\infty))$ and a well-defined Riemann tensor.

Surprisingly there is no need to impose cylindrical symmetry on the solution, since we have a regular axis at $r = 0$ provided that f, g are regular there and we have already required it. According to [4], the axis is regular if

$$\lim_{r \to 0} \frac{\langle \operatorname{grad}\Delta, \operatorname{grad}\Delta \rangle}{4\Delta} = e^{2(U-K)}|_{r=0} = 1, \quad \Delta = \langle \partial_\phi, \partial_\phi \rangle = r^2 e^{2U}, \quad (10)$$

therefore we need that $K(0,t) = U(0,t)$ for every value of t. But at the axis the equations (7a,7b) that determine K are rather simple,

$$K_r = U_r, \qquad K_t = U_t, \quad (11)$$

and therefore $K(0,t) = U(0,t) + \text{const.}$ and the condition of regularity at the axis is fulfilled by either taking the constant of integration equal to zero or conveniently rescaling the angular coordinate.

Note that this requirement of regularity excludes a timelike gradient of ρ in the vicinity of the axis.

III.   GEODESIC COMPLETENESS

The metric that is obtained after integrating the system (7a,7b) has regular components in the whole spacetime, but this not suffices in order to have a non-singular spacetime. We shall consider that a spacetime is regular [7] if it is causally geodesically complete, that is, if every causal geodesic may be extended to all values of its affine parametrization.

This means analysing the geodesic equations for diagonal cylindrically symmetric spacetimes. This has been done in [8]. Those results can be summarized as follows:

Theorem: A cylindrically symmetric diagonal metric in the form (1) with C^2 metric functions f, g, ρ is future causally geodesically complete provided that along causal geodesics:

    1. For large values of t and increasing r,
        (a) $K_r + K_t \geq 0$, and either $K_r \geq 0$ or $|K_r| \lesssim K_r + K_t$.
        (b) $(K+U)_r + (K+U)_t \geq 0$, and either $(K+U)_r \geq 0$ or $|(K+U)_r| \lesssim (K+U)_r + (K+U)_t$.
        (c) $(K - U - \ln\rho)_r + (K - U - \ln\rho)_t \geq 0$, and either $(K - U - \ln\rho)_r \geq 0$ or $|(K - U - \ln\rho)_r| \lesssim (K - U - \ln\rho)_r + (K - U - \ln\rho)_t$.

2. For large values of t, constant b exist such that $\left.\begin{array}{c} K(t,r) - U(t,r) \\ 2\,K(t,r) \\ K(t,r) + U(t,r) + \ln\rho(t,r) \end{array}\right\} \geq -\ln|t| + b.$

A similar result can be stated for past-pointing geodesics just reversing the sign of the time derivatives in condition 1.

Note that we have omitted the condition for non-radial geodesics with decreasing r in [8], since according to (97) in that reference, non-radial causal geodesics would reach $r = 0$ for finite t, that is, before t becomes singular, contradicting the fact that geodesics should be singular there. Therefore the axis cannot be reached by geodesics with non-zero angular momentum.

All we have to do now in order to have a geodesically complete model is to check whether conditions 1 and 2 are satisfied.

We first show that the conditions on the derivatives are always fulfilled for a stiff fluid model:

1. (a) According to (7a,7b) we have,

$$K_t + K_r = U_t + U_r + r(U_t + U_r)^2 + \alpha r.$$

Several possibilities are open: When $U_t + U_r$ is positive, $K_t + K_r$ is positive. If $U_t + U_r$ is negative and $|U_t + U_r| \geq 1$, the quadratic term is larger and $K_t + K_r$ is again positive for large values of r. Finally, if $U_t + U_r$ is negative and $|U_t + U_r| \leq 1$, it is the pressure term αr which overcomes the negative term for large r.

The same sort of reasoning is valid to conclude that the radial derivative,

$$K_r = U_r + r(U_t^2 + U_r^2) + \alpha r.$$

is positive for large values of r.

(b) We can apply the same argument to $K + U$,

$$(K + U)_t + (K + U)_r = 2(U_t + U_r) + r(U_t + U_r)^2 + \alpha r,$$

in order to show that these derivatives and $(K + U)_r$ are positive for large values of the radial coordinate.

(c) Finally, the third condition is always satisfied,

$$(K - U - \ln\rho)_t + (K - U - \ln\rho)_r = r(U_t + U_r)^2 - \frac{1}{r} + \alpha r \geq 0,$$

$$(K - U - \ln\rho)_r = r(U_t^2 + U_r^2) - \frac{1}{r} + \alpha r \geq 0,$$

for increasing r.

Past-pointing geodesics are treated analogously without any additional problem, since reversing the sign of time derivatives does not alter the positivity of the quadratic terms.

Consequently only condition 2. yields a restriction.

2. As we shall see, this condition amounts to study U at the axis for large values of the time coordinate:

(a) This condition is trivial since $(K - U)|_{r=0} = 0$,

$$\left(K - U\right)(t, r) = \int_0^r dr'\,(K - U)_r(t, r') = \int_0^r dr'\,\left\{r'\left(U_t^2(t, r') + U_r^2(t, r')\right) + \alpha r\right\} > 0.$$

(b) The previous reasoning for ruling out singularities for decreasing radius leaves us with two possibilities: increasing radius and constant radius, $r = 0$. Since, according to (11)

$$K(r, t) = U(0, t) + \int_0^r dr'\,K_r(r', t),$$

and we have already checked that K_r is positive for large t and increasing r, we just have to study the term $U|_{r=0}$, that is, we have reduced the problem to analysing the behaviour of K at the axis for large values of t.

(c) Similarly,

$$\big(K+U\big)(r,t) + \ln r = 2U(0,t) + \ln r + \int_0^r dr' \, \big(K+U\big)(r',t),$$

and we have already checked the positivity of $K_r + U_r$ as in the previous condition. The logarithmic term does not mean a problem for increasing radius. Again we are left with controlling the behaviour of the U term.

Summarizing our results so far, in order to have a causally geodesically complete spacetime we just have to require that $U|_{r=0}$ does not decrease faster than a negative logarithm for large values of the absolute value of the time coordinate. The condition on the solution of the Cauchy problem for the wave equation at the axis becomes a bit simpler,

$$K(0,t) = U(0,t) = \int_0^1 d\tau \frac{\tau}{\sqrt{1-\tau^2}} \{tg(|t|\tau) + f(|t|\tau) + |t|\tau f'(|t|\tau)\} \geq -\frac{1}{2}\ln|t| + b, \tag{12}$$

since the dependence on the polar angle is lost.

This bound can be attained, for example, by the initial data,

$$f(r) = \frac{|a|-1}{2}\ln r, \qquad g(r) = \frac{a}{\pi}\frac{\ln r}{r},$$

since for this choice of functions the solution to the Cauchy problem for the wave equation is,

$$U(0,t) = \frac{1}{2}\big(|a| + a\operatorname{sign}t - 1\big)\ln|t| + \frac{1}{2}\big(|a| - a\operatorname{sign}t - 1\big)\ln 2 = \begin{cases} \frac{1}{2}\big(2|a|-1\big)\ln|t| - \ln 2 & \operatorname{sign}at > 0 \\ \frac{1}{2}\big(2|a|-1\big)\ln 2 - \frac{1}{2}\ln|t| & \operatorname{sign}at < 0 \end{cases}.$$

The behaviour of the terms in (12) is rather different. The term, U_f, dependent on f, the initial value of U, is even in the time coordinate, as it is to be expected when the initial time derivative of U is zero. On the contrary, the term U_g, dependent on g, is odd in t.

This means that if U_f satisfies (12) for positive time, it is automatically satisfied for negative time. On the contrary, if U_g satisfies (12) for positive time, it is only satisfied for negative time if it is also satisfied by U_{-g} for positive time. Therefore three different possibilities are open depending on the value of

$$\lim_{r\to\infty}\frac{f(r) + rf'(r)}{rg(r)}. \tag{13}$$

- If (13) is infinite, we need $U_f(t) > -\frac{1}{2}\ln|t| + b$ for large values of t in order to have geodesic completeness. This means that $f(r) + rf'(r) > -\frac{1}{2}\ln r + k$ for large r.

- If (13) is zero, we need $|U_g(t)| < \frac{1}{2}\ln|t| + b$ for large values of t in order to have geodesic completeness. This means that $|g(r)| < \frac{1}{\pi r}\ln r + k$ for large r.

- If (13) is finite, then U_f and U_g are of the same order for large values of t or of $-t$ and geodesic completeness will depend on the value of the limit.

Note that values of the integrand close to $\tau = 0$ do not influence the result for large values of t, since we may split the integral in two terms,

$$U(0,t) = \int_0^{x_c/t} + \int_{x_c/t}^1,$$

and the first one is bounded and negligible for large t. Therefore the main contribution to U comes from the second term, which must fulfill the required asymptotic behaviour.

IV. EXAMPLES

A simple and wide family of functions that satisfy (12) can be written in terms of polynomials. Consider

$$f(r) = \sum_{i=0}^{n} a_i r^i, \qquad g(r) = \sum_{i=0}^{m} b_i r^i. \tag{14}$$

If n, m are even numbers, U can be analytically integrated in terms of polynomials. For our purposes we just require U at the axis, which can be integrated for a larger set of functions. Since U is linear in f and g, we may analyse the monomials independently. For $f(r) = r^n$, $g(r) = r^m$ we obtain,

$$U_f(t) = \frac{n!!}{(n-1)!!} \left(\frac{\pi}{2}\right)^{(1+(-1)^{n+1})/2} |t|^n, \qquad U_g(t) = \frac{m!!}{(m+1)!!} \left(\frac{\pi}{2}\right)^{(1+(-1)^{m+1})/2} |t|^m t. \tag{15}$$

These expressions are valid even for $n, m = -1$ taking $(-1)!! := 1$, although they may be not very practical.

According to (12) we have two different possibilities for obtaining a singularity-free model:

- If f, g are polynomials in r respectively of degree n, m and $n > m + 1$, we have a non-singular model if a_n is positive.

- If f, g are polynomials in r respectively of degree $n, n - 1$, U_f and U_g at the axis are polynomials of degree n and we have a non-singular model if U_f dominates over U_g. This happens if the leading term of U_f is greater than the one of U_g, that is,

$$\left(\frac{n!!}{(n-1)!!}\right)^2 \left(\frac{2}{\pi}\right)^{(-1)^n} \frac{a_n}{|b_{n-1}|} > 1. \tag{16}$$

Using Stirling's formula for approximation of factorials, an easy and safe bound would be

$$\left(n + \frac{1}{2}\right) a_n > |b_{n-1}|. \tag{17}$$

This family of non-singular cosmological models is large indeed, as it can be seen by restricting the range to a finite dimensional space of polynomial functions: if we consider the space of functions U for which $U|_{r=0}$ is a polynomial of degree equal or lower than n, the subset of singularity-free models comprises an open set, according to (17).

This result can be generalized, since

$$\int_0^1 d\tau \frac{\tau^{p+1}}{\sqrt{1-\tau^2}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma((p+2)/2)}{\Gamma((p+3)/2)} \tag{18}$$

allows integration for every real value of the exponent p. Therefore, for $f(r) = r^p$, $g(r) = r^q$, we obtain,

$$U_f(t) = \sqrt{\pi} \frac{\Gamma((p+2)/2)}{\Gamma((p+1)/2)} |t|^p, \qquad U_g(t) = \frac{\sqrt{\pi}}{2} \frac{\Gamma((q+2)/2)}{\Gamma((q+3)/2)} |t|^q t, \tag{19}$$

which allow generalization of the geodesic completeness requirements which have been previously derived for polynomial functions to linear combinations of powers of r. Additionally one has just to care about the class of differentiability of U, which demands that $p \geq 2$, $q > 2$.

V. CONCLUSIONS

We have analysed a wide family of stiff perfect fluid cosmological models with cylindrical symmetry. The issue of causal geodesic completeness has been reduced to just the behaviour at the axis of the initial value problem for a sourceless 2D-wave equation, which is the only one left after simplifying Einstein equations. A sufficient condition for geodesic completeness is provided, which is very easy to check and to implement. The case of polynomial initial data

has been discussed and allows a fairly large set of non-singular cosmological models. We think that this set is wide enough to preclude considering non-singular models as isolated points in a space of cosmological models.

The role of pressure in these models is obviously determinant, since stiff perfect fluids are a limit case for energy conditions, corresponding to a sound velocity equal to that of light. On the contrary, dust perfect fluids are always singular according to Raychaudhuri equation. Intermediate cases remain open for discussion, even though partial results have been obtained [3].

It is interesting to notice that nonseparability of the models in these coordinates is fundamental for geodesic completeness. In [9] separable cosmologies were studied and *none* of them was found to be regular.

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